Covariant Symplectic Structure and Conserved Charges of NMG

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December 27, 2011

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Definition of Symplectic Structure

- Consider a smooth manifold Z endowed with a 2-form given as
$$\omega = dp_i \wedge dq^i, \tag{1}$$
where q^i, p_i are coordinates and momenta, $i = 1, \ldots N$.
- $$\omega = \frac{1}{2}\omega_{IJ} dQ^I \wedge dQ^J, \tag{2}$$
with $I = 1, ..., 2N$. $Q^i = p_i$ for $i \leq N$ and $Q^i = q^{i-N}$ for $i \geq N$. Then, $\omega_{i,i+N} = -\omega_{i+N,i} = -1$
- Poisson bracket of any two function $A(Q)$ and $B(Q)$ is given by
$$[A, B] = \omega^{IJ} \frac{\partial A}{\partial Q^I} \frac{\partial B}{\partial Q^J}. \tag{3}$$

Definition of Symplectic Structure

- ω is closed ($d\omega = 0$)
- nondegenerate, i.e. when ω is written as a $2N \times 2N$ matrix, it has an inverse.
- This closed 2-form on Z is called symplectic structure.

Symplectic Structure in Geometrical Theories

- Choosing p_i, q^i as coordinates of the phase space Z would destroy the general covariance (by the choice of time coordinate).
- One should construct the phase space Z from solutions of the equations of motion derived from an action to achieve a manifestly covariant structure.
- Since classical solutions of any physical theory is in one-to-one correspondence with the initial values of p_i, q^i, we define our phase space as the space of solutions of the classical equations. (Crnkovic,Witten-1987)

Fundamental Objects on Z - Functions

- Let g be a solution of the field equations Φ.
- The functions on Z , denoted by $g(x)$, takes a spacetime point x and maps it into a $D \times D$ real matrix $g(x)$.

Fundamental Objects on Z - Vectors

- Consider an arbitrary, small variation in the metric $\tilde{g} = g + \delta g$
- When inserted into the field equations, it yields $\tilde{\Phi} = \Phi + \delta\Phi$.
- The vectors can be defined as the variations δg which solve $\delta\Phi = 0$ (preserves the field equations).

Fundamental Objects on Z - Forms

- A one-form, $\delta g(x)$, maps a vector δg to a $D \times D$ real matrix $\delta g(x)$, which is the vector evaluated at a spacetime x.
- We generalize this notion to construct p-forms as "wedge functions" of the one forms $\delta g(x)$

$$\Omega = \int dx_1 \cdots dx_p \, \Theta(x_1, \cdots, x_p) \, \delta g(x_1) \wedge \cdots \wedge \delta g(x_p), \quad (4)$$

where $\Theta(x_1, \cdots, x_p)$ is a zero-form on Z and $\wedge$ is an anticommuting product.

Fundamental Objects on Z - Forms

- We can define an exterior derivative operator δ which maps p-forms to p+1-forms as follows

$$\delta\Omega = \int dx_0\, dx_1 \cdots dx_p \, \frac{\delta\Theta(x_1, \cdots, x_p)}{\delta g(x_0)} \, \delta g(x_0) \wedge \delta g(x_1) \wedge \cdots \wedge \delta g(x_p). \tag{5}$$

- One can easily check this operator obeys the modified Leibniz rule and Poincaré Lemma.

Application to a Local Gravity Action

- Let us consider a generic local gravity action

$$S = \int d^D x \, \sqrt{|g|} \, \mathcal{L}(g, R, \nabla R, R^2, \cdots), \tag{6}$$

whose variation is given as

$$\delta S = \int d^D x \, \sqrt{|g|} \, \Phi_{ab} \delta g^{ab} + \int d^D x \, \partial_a \Lambda^a (g, \delta g, \nabla \delta g \cdots), \tag{7}$$

where Φ_{ab} is the field equation and Λ^a is the boundary term.

Application to a Local Gravity Action

- We can view δS as a 1-form on Z (note that $\Lambda^a(x)$ includes δg_{ab} and the relevant quantities).
- The exterior derivative of (7) will vanish by Poincaré Lemma,

$$\delta^2 S = \int d^D x \, \sqrt{|g|}\, \delta\Phi_{ab} \wedge \delta g^{ab} - \frac{1}{2} \int d^D x \, \sqrt{|g|}\, \Phi_{ab}\, \delta g^{ab} \wedge \delta \ln|g| + \int d^D x \, \partial_a \delta\Lambda^a = 0. \tag{8}$$

where $\delta \ln|g| = g^{ab}\delta g_{ab} = -g_{ab}\delta g^{ab}$.
- First two integrals vanish on shell and the third one implies

$$\int d^D x \, \sqrt{|g|}\, \nabla_a J^a = 0, \tag{9}$$

where $J^a \equiv -\frac{\delta\Lambda^a}{\sqrt{|g|}}$ is the "symplectic current".

Application to a Local Gravity Action

- From this, one can construct the following Poincaré invariant 2-form since the covariant divergence of the symplectic current vanishes ($\nabla_a J^a = 0$)

$$\omega = \int_\Sigma d\Sigma_a \sqrt{|g|}\, J^a \,, \tag{10}$$

  where Σ is $(D-1)$-dimensional spacelike hypersurface.
- Darboux's theorem assures us that this is the sought after symplectic structure of the theory if ω is additionally closed.

Application to a Local Gravity Action

- Finally, one must show that it is also gauge invariant in the space of classical solutions Z and in the quotient space $\bar{Z} = Z/G$, G being group of diffeomorphisms ($x^a \rightarrow x^a + \xi^a$).
- The former is trivial since all constituents of ω transform like tensors. For the latter, we should find out how ω transforms under the following transformation

$$\delta g_{ab} \rightarrow \delta g_{ab} + \nabla_a \xi_b + \nabla_b \xi_a \tag{11}$$

where ξ is asymptotic to a Killing vector field at infinity. This computation will yield a boundary term which gives rise to conserved charges of the theory under consideration.

Application to a Local Gravity Action

- Let us now apply this procedure to the following quadratic action

$$I = \int d^D x \, \sqrt{|g|} \, \mathcal{L} \equiv \int d^D x \, \sqrt{|g|} \left(\frac{1}{\kappa}(R + 2\Lambda_0) + \alpha R^2 + \beta R_{ab}^2 \right) \tag{12}$$

- One can explicitly show that covariant divergence of the symplectic current is equal to the following expression

$$\nabla_a J^a = \frac{1}{2} g^{ab} \delta\Phi_{ab} \wedge \delta \ln|g| + \delta\Phi_{ab} \wedge \delta g^{ab} + \delta\Phi \wedge \delta \ln|g|, \tag{13}$$

which vanishes on shell. Here $\Phi_{ab} \equiv \frac{1}{\kappa}\mathcal{G}_{ab} + \alpha A_{ab} + \beta B_{ab}$ and $\Phi = g^{ab}\Phi_{ab}$.

Application to a Local Gravity Action

- It can also be shown, without using the field equations, that ω is a closed form. We have

$$\delta\omega = \int_{\Sigma} d\Sigma_a \,(\delta\sqrt{|g|} \wedge J^a + \sqrt{|g|}\,\delta J^a), \tag{14}$$

  and variation of the current reads

$$\delta J^a = -\frac{1}{2} J^a \wedge \delta \ln|g|. \tag{15}$$

  By virtue of (15) and bearing in mind that J^a is anticommuting 2-form, (14) vanishes.

Application to a Local Gravity Action

- There remains to investigate the gauge invariance of ω. After a cumbersome calculation, change in the symplectic current can be written as

$$\Delta J^a = \nabla_c \mathcal{F}^{ac} + g^{bc} \delta\Phi_{bc} \wedge \xi^a + 2\Phi_{bc}\, \xi^c \wedge \delta g^{ab} + \Phi^{ac}\, \xi_c \wedge \delta \ln|g| + \xi^a \wedge \delta\Phi \quad (16)$$

- First term in (16) vanishes when inserted in the integral for ω for sufficiently fast decaying metric variations, the remaining terms vanish on-shell.

Application to a Local Gravity Action

where

$$\mathcal{F}^{ac} = -\mathcal{F}^{ca} = \frac{1}{\kappa}\mathcal{F}_{\kappa}^{ac} + \alpha\mathcal{F}_{\alpha}^{ac} + \beta\mathcal{F}_{\beta}^{ac}\,, \tag{17}$$

with

$$\begin{aligned} \mathcal{F}_{\kappa}^{ac} \equiv\; & 2\xi^{[c} \wedge \nabla_b \delta g^{a]b} - 2\xi_b \wedge \nabla^{[c}\delta g^{a]b} - 2\delta g^{b[c} \wedge \nabla_b \xi^{a]} \\ & -2\xi^{[a} \wedge \nabla^{c]}\delta \ln|g| - \delta \ln|g| \wedge \nabla^{[c}\xi^{a]} \end{aligned} \tag{18}$$

Calculation of Conserved Charges

- We linearize the metric as $g_{ab} = \bar{g}_{ab} + h_{ab}$
- Indices are raised/lowered and covariant derivatives are defined with respect to the background metric $\bar{g}_{ab}$ as usual.
- One should take the diffeomorphisms as the isometries of the background spacetime, meaning $\bar{\nabla}_a \bar{\xi}_b + \bar{\nabla}_b \bar{\xi}_a = 0$.
- Assume the background spacetime $\bar{g}_{ab}$ admits a globally defined Killing vector $\bar{\xi}_a$.
- The variation is identified as $\delta g_{ab} \to h_{ab}$, $\delta g^{ab} \to -h^{ab}$. Therefore, the terms R_{ab}, R are identified with the background ones $\bar{R}_{ab}$, $\bar{R}$ and terms like $\delta(\nabla_a R_{bc})$ are taken as $(\nabla_a R_{bc})_L$, where subscript L means linearized version of the corresponding quantity.
- Finally, we write the ξ terms at the right hand side of the wedge products and then drop them.

Calculation of Conserved Charges

- With all these identifications the relevant charge expression is given by

$$
\begin{aligned}
Q(\bar{\xi}) &= \frac{1}{2}\int_{\Sigma} d^{D-1}x\ \sqrt{|\sigma|}\, n_a\, \bar{\nabla}_c Q^{ac} \\
&= \frac{1}{2}\int_{\partial\Sigma} d^{D-2}x\ \sqrt{|\sigma^{(\partial\Sigma)}|}\, n_a\, s_c\, Q^{ac}\,,
\end{aligned}
\qquad (19)
$$

where Σ is a $(D-1)$-dimensional spacelike hypersurface with induced metric σ and unit normal vector n^a, $\partial\Sigma$ (boundary of Σ) is a $(D-2)$-dimensional hypersurface with induced metric $\sigma^{(\partial\Sigma)}$ and unit normal s^c.

Calculation of Conserved Charges

$$Q^{ac} = -Q^{ca} = \frac{1}{\kappa} Q^{ac}_{\kappa} + \alpha Q^{ac}_{\alpha} + \beta Q^{ac}_{\beta}, \qquad (20)$$

with

$$Q^{ac}_{\kappa} \equiv 2\bar{\nabla}_b h^{b[a}\bar{\xi}^{c]} - 2\bar{\nabla}^{[c} h^{a]b}\bar{\xi}_b - 2h^{b[c}\bar{\nabla}_b\bar{\xi}^{a]} + 2(\bar{\nabla}^{[c} h)\bar{\xi}^{a]} - h\bar{\nabla}^{[c}\bar{\xi}^{a]} \qquad (21)$$

$$\begin{aligned} \mathcal{F}^{ac}_{\kappa} \quad \equiv \quad & 2\xi^{[c} \wedge \nabla_b \delta g^{a]b} - 2\xi_b \wedge \nabla^{[c}\delta g^{a]b} - 2\delta g^{b[c} \wedge \nabla_b \xi^{a]} \\ & -2\xi^{[a} \wedge \nabla^{c]}\delta \ln|g| - \delta \ln|g| \wedge \nabla^{[c}\xi^{a]} \end{aligned}$$

(22)

Calculation of Conserved Charges

- Charge expressions given above are identical to the ones given in the literature.
- Remember that we assumed $\delta\phi_{ab} = 0$ while constructing the vectors. Here, this condition implies $(\Phi_{ab})_L = 0$ at the boundary of the spacetime.

Calculation of Conserved Charges - Examples

- 3-dimensional Lifshitz Blackhole

$$ds^2 = -\frac{r^6}{\ell^6}\left(1 - \frac{M\ell^2}{r^2}\right)dt^2 + \frac{\ell^2}{r^2}\left(1 - \frac{M\ell^2}{r^2}\right)^{-1} dr^2 + \frac{r^2}{\ell^2}dx^2, \quad (23)$$

$$\Lambda_0 = \frac{13}{2\ell^2}, \quad \beta = \frac{2\ell^2}{\kappa}, \quad \alpha = -\frac{3\ell^2}{4\kappa}, \quad \kappa = 16\pi G$$

with background

$$ds^2 = -\frac{r^6}{\ell^6}dt^2 + \frac{\ell^2}{r^2}dr^2 + \frac{r^2}{\ell^2}dx^2,$$

Calculation of Conserved Charges - An Example

$$n^a = -\frac{\ell^3}{r^3}\delta^a_t,\ s^a = \frac{r}{l}\delta^a_r$$

$$E = \lim_{r\to\infty}\int_0^{2\pi\ell} \frac{r^3}{\ell^3}\, n_t\, s_r\, Q^{tr}(\bar{\xi})dx = \frac{7m^2}{4G}, \tag{24}$$

$$(\Phi_{ab})_L = \frac{6M^3}{\ell^2} \tag{25}$$

at the boundary.

Conclusion

- We have constructed the symplectic structure of NMG and calculated the conserved charges of some solutions using the diffeomorphism invariance of the symplectic two-form.
- This method gives rise to the condition $(\Phi_{ab})_L = 0$ at the boundary, which the solutions with *AdS* background satisfy.
- However, solutions with arbitrary background which we have considered do not satisfy the condition $(\Phi_{ab})_L = 0$ at the boundary. This might be the reason for the discrepancy in the thermodynamics of these blackholes.
(Devecioglu,Sarioglu-2011)