Einstein-İnfeld-Hoffmann Aproach
For Strings

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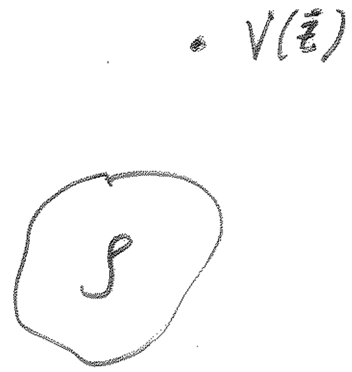
Newton's Gravity



- $$m \frac{d^2 \vec{x}}{dt^2} = - \vec{\nabla} V$$

- $$\nabla^2 V = 4\pi \rho$$

Maxwell's Theory

- $$\partial_\mu F^{\mu\nu} = 4\pi J^\nu$$

- $$m \frac{d^2 x^\mu}{dt^2} = e\, F^\mu{}_\nu \frac{dx^\nu}{dt}$$

(B)

Einstein's Theory

$G_{\mu\nu} = \alpha T_{\mu\nu}$ "field Eqns"

$$\frac{d^2 z^\mu}{ds^2} + \Gamma^\mu_{\alpha\beta} \frac{dz^\alpha}{ds} \frac{dz^\beta}{ds} = 0$$ "Geodesic equation"

- Einstein's Theory is the only one "geodesic equations" or "equations of motion" are derivable from the Field Equations

1 A. Eienstein, L. Infeld and B. Hoffmann, Ann. Math. 30 (2) (1938).

2. B. Adler, M. Bazin and M.M. Schiffer, Introduction to General Relativity, (Mc Graw-Hill), New York, 1965), p. 296

3. M. Gürses and F. Gürsey, Physical Review D 11, 967 (1975).

The field equations are

$$G_{\mu\nu} = \kappa T_{\mu\nu} \qquad (1)$$

From the Bianchi identities

$$\nabla_\mu T^{\mu\nu} = 0$$

or

$$\partial_\mu \left(\sqrt{-g}\, T^{\mu\nu} \right) + \Gamma^{\nu}_{\alpha\beta} \sqrt{-g}\, T^{\alpha\beta} = 0$$

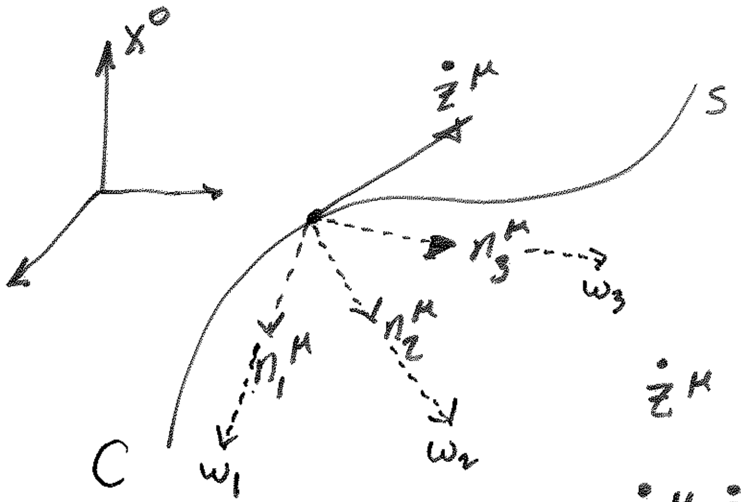
The geodesic equations (motion of free particles)

$$\frac{d^2 z^\mu}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dz^\alpha}{ds} \frac{dx^\beta}{ds} = 0 \qquad (2)$$

Einstein Infeld Hoffmann (EIH) shows that (1) and (2) are not independent. (2) follows from (1)

1. For a point particle

$$\sqrt{-g}\, T^{\mu\nu} = \varepsilon \int \frac{dz^\mu}{ds}\frac{dz^\nu}{ds}\, \delta^{(4)}(x - z(s))\, ds$$



$$ds^2 = g_{\mu\nu}\, dz^\mu dz^\nu$$

$$\dot{z}^\mu g_{\mu\nu} n_i^\nu = 0\ , \quad i = 1, 2, 3$$

$$\dot{z}^\mu \dot{z}^\nu g_{\mu\nu} = 1$$

$C:\quad x^\mu = z^\mu(s) = f^\mu(s, 0, 0, 0).$

For points not on C

$$x^\mu = f^\mu(s, \omega_1, \omega_2, \omega_3) = f^\mu(\omega^\mu)$$

with $\quad \omega^\mu = (s, \omega_1, \omega_2, \omega_3)$

Then

$$z^\mu(s) = \iiint f^\mu(s, \omega_1, \omega_2, \omega_3)\, \delta^3(\omega)\, d^3\omega$$

Then we have

$$\sqrt{-g}\, T^{\mu\nu} = \varepsilon \int \dot{z}^\mu \dot{z}^\nu \,\delta^{(4)}(x^\alpha - f^\alpha(\omega))\, \delta^3(\omega)\, d^3\omega\, ds$$

$$= \varepsilon \int \dot{z}^\mu \dot{z}^\nu \,\delta^{(4)}(x - f(\omega))\, \delta^3(\omega)\, d^4\omega$$

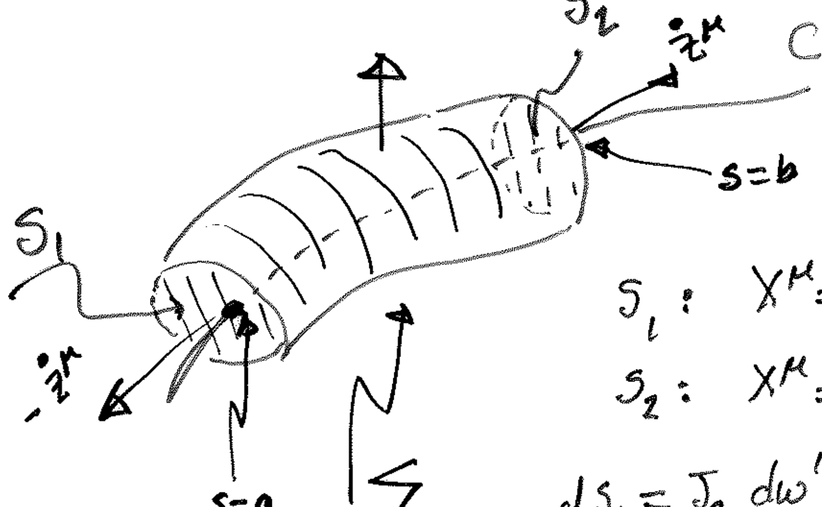
coordinate transformations $x^\mu = f^\mu(\omega)$

$$d^4x = J\, d^4\omega$$

Then we obtain

$$\sqrt{-g}\, T^{\mu\nu} = \varepsilon\, \dot{z}^\mu \dot{z}^\nu \,\frac{1}{J}\, \delta^3(\omega)$$

Integrating the energy conservation in a tubular region Σ (metric is independent of s)



S_1: $X^\mu = f^\mu(a, \omega^1, \omega^2, \omega^3)$

S_2: $X^\mu = f^\mu(b, \omega^1, \omega^2, \omega^3)$

$dS_1 = J_a\, d\omega^1 d\omega^2 d\omega^3, \quad J_a = J|_{s=a}$

$dS_2 = J_b\, d\omega^1 d\omega^2 d\omega^3, \quad J_b = J_{s=b}$

$$\int_{\Sigma} \partial_\mu (+\sqrt{-g}\, T^{\mu\nu})\, d\Sigma + \int_{\Sigma} \Gamma^{\nu}_{\alpha\beta} \sqrt{-g}\, T^{\alpha\beta} d\Sigma = 0$$

$$\underbrace{\int_{\partial\Sigma} \sqrt{-g}\, T^{\mu\nu} dS_\mu}_{I^\nu} + \underbrace{\int_{\Sigma} \Gamma^{\nu}_{\alpha\beta} \sqrt{-g}\, T^{\alpha\beta} d\Sigma}_{J^\nu} = 0$$

$$I^\nu \equiv \int_{\partial\Sigma} \sqrt{-g}\, T^{\mu\nu} dS_\mu = \int_{S} \sqrt{-g}\, T^{\mu\nu} N_\mu\, dS$$

$$+ \int_{S_2} \sqrt{-g}\, T^{\mu\nu} \dot{z}_\mu\, dS_2 - \int_{S_1} \sqrt{-g}\, T^{\mu\nu} \dot{z}_\mu\, dS_1$$

$$= \varepsilon \int_{S_2} \dot{z}^\nu \Big|_{s=b} \frac{1}{J_b} \delta^3(\omega)\, dS_2 - \varepsilon \int_{S_1} \dot{z}^\nu \Big|_{s=a} \frac{1}{J_a} \delta^3(\omega)\, dS_1$$

$$= \varepsilon \left(\dot{z}^\nu \Big|_{s=b} - \dot{z}^\nu \Big|_{s=a} \right) = \varepsilon \int_a^b \frac{d}{ds} \dot{z}^\nu\, ds$$

$$I^\nu = \varepsilon \int_a^b \frac{d^2 z^\nu}{ds^2}\, ds$$

$$\frac{dS_2}{J_b} = d^3\omega$$

$$\frac{dS_1}{J_a} = d^3\omega$$

$$J^{\nu} \equiv \int_{\Sigma} \Gamma^{\nu}_{\alpha\beta} \sqrt{-g}\, T^{\alpha\beta}\, d\Sigma$$

$$= \varepsilon \int_{\Sigma} \int ds\, \Gamma^{\nu}_{\alpha\beta}\, \dot{z}^{\alpha} \dot{z}^{\beta}\, \delta^{(4)}(x-z)\, d\Sigma$$

$$= \varepsilon \int_{a}^{b} ds\, \Gamma^{\nu}_{\alpha\beta}\, \dot{z}^{\alpha} \dot{z}^{\beta}$$

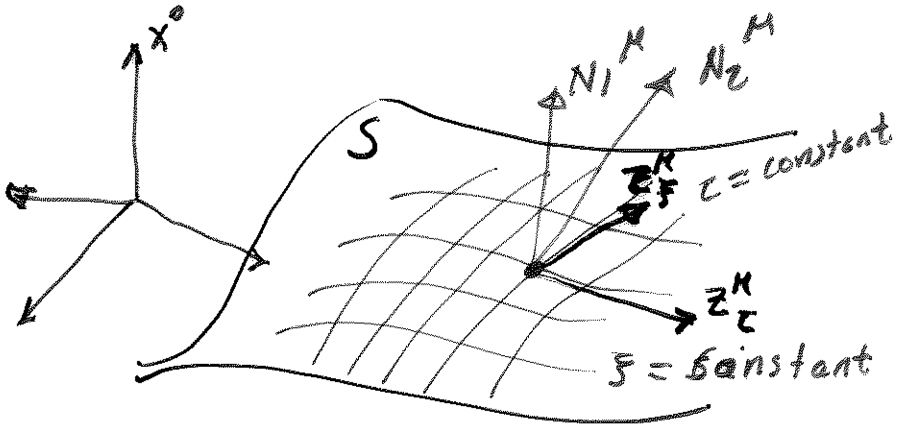
$$I^{\nu} + J^{\nu} = 0 \Rightarrow$$

$$\frac{d^2 z^{\mu}}{ds^2} + \Gamma^{\mu}_{\alpha\beta} \frac{dz^{\alpha}}{ds} \frac{dz^{\beta}}{ds} = 0$$

" The geodesic equation "

' The effect of the point particle to the geometry is ignored "

Strings in GR



S = "world sheet"

$$S: \quad X^\mu = z^\mu(\tau, \xi)$$

Normal vectors $\{ N_1^\mu, N_2^\nu \}$

tangent vectors $\left\{ u^\mu = \frac{\partial z^\mu}{\partial \tau}, \; v^\mu = \frac{\partial z^\mu}{\partial \xi} \right\}$

$$\{ N_1^\mu, N_2^\mu \} \perp \{ u^\mu, v^\mu \}$$

Define a bivector $\sigma_{\mu\nu}$

$$\sigma^{\alpha\beta} = u^\alpha v^\beta - u^\beta v^\alpha$$

Nambu functional for string equation of motion

$$\delta \iint d\tau \, d\xi \, (\sigma_{\alpha\beta} \sigma^{\beta\alpha})^{1/2} = 0$$

Consider now a coordinate transformation

$$X^\mu = f^\mu(\tau, \omega^1, \omega^2, \xi)$$

$$[\ \xi^\mu = (\tau, \omega^1, \omega^2, \xi) = F^\mu(X^\alpha)\ ]$$

New coordinates are choosen so that

$$S: \quad \omega^1 = 0, \quad \omega^2 = 0$$

Hence

$$Z^\mu(\tau, \xi) = f^\mu(\tau, 0, 0, \xi)$$

Then we get

$$\sqrt{-g}\, T^{\mu\nu} = \iint H^{\mu\nu}\, \delta^4(X^\alpha - f^\alpha(\tau, 0, 0, \xi))\, d\tau\, d\xi$$

$$= \iiint H^{\mu\nu}(\tau, \xi)\, \delta^4(X^\alpha - f^\alpha(\tau, \omega^1, \omega^2, \xi))\, \delta^2(\omega)\, d\omega^1 d\omega^2 d\tau d\xi$$

$$= \iiint H^{\mu\nu}(\tau, \xi)\, \delta^4(X - f)\, \delta^2(\omega)\, d^4\xi$$

$$J\, d^4\xi = d^4 X$$

$$\sqrt{-g}\, T^{\mu\nu} = \frac{1}{J} H^{\mu\nu}(\tau, \xi)\, \delta^2(\omega).$$

similar to one particle energy momentum tensor we define a single string energy momentum tensor

$$\sqrt{-g}\, T^{\mu\nu} = \iint \frac{\sigma^{\mu\alpha}\sigma^{\nu}{}_{\alpha}}{\left(\frac{1}{2}\sigma^{\beta\gamma}\sigma_{\gamma\beta}\right)^{1/2}} \delta^{(4)}(x - z(\tau,\xi))\, d\tau\, d\xi$$

Define

$$H^{\mu\nu} = \frac{\sigma^{\mu\alpha}\sigma^{\nu}{}_{\alpha}}{\left(\frac{1}{2}\sigma^{\beta\gamma}\sigma_{\gamma\beta}\right)^{1/2}} = u^{\mu}v^{\nu} - u^{\nu}v^{\mu}$$

Here we assumed that

$$g_{\mu\nu}\, dz^{\mu} dz^{\nu} = g_{\mu\nu}\left(\frac{\partial z^{\mu}}{\partial \tau} d\tau + \frac{\partial z^{\mu}}{\partial \xi} d\xi\right)\left(\frac{\partial z^{\nu}}{\partial \xi} d\tau + \frac{\partial z^{\nu}}{\partial \xi} d\xi\right)$$

$$= u^2 d\tau^2 + 2u\cdot v\, d\tau\, d\xi + v^2 d\xi^2$$

We can chooze

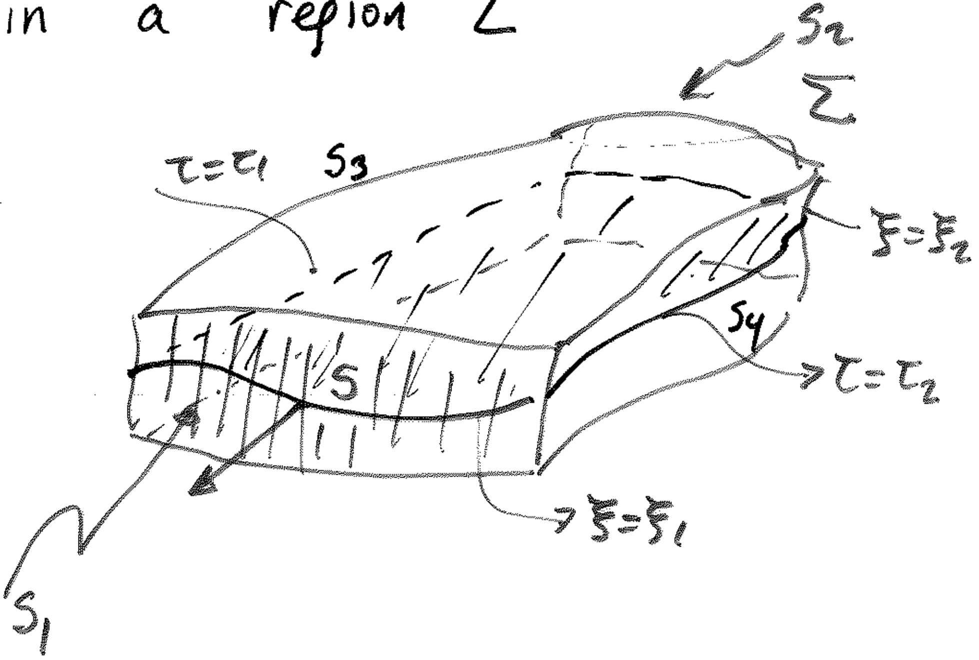
$$u^2 + v^2 = 0 \quad , \quad u\cdot v = 0$$

Metric on S : $ds^2 = u^2(d\tau^2 - d\xi^2)$

Integrating

$$\partial_\mu \left(\sqrt{-g}\, T^{\mu\nu} \right) + \Gamma^{\nu}_{\alpha\beta} \sqrt{-g}\, T^{\alpha\beta} = 0$$

in a region Σ



$$\iint_{\Sigma} \partial_\mu \left(\sqrt{-g}\, T^{\mu\nu} \right) d\Sigma = \int_{\partial\Sigma} \sqrt{-g}\, T^{\mu\nu} dS_\mu$$

$$= \int_{S_1} \sqrt{-g}\, T^{\mu\nu} dS_\mu + \int_{S_2} \sqrt{-g}\, T^{\mu\nu} dS_\mu$$

$$+ \int_{S_3} \sqrt{-g}\, T^{\mu\nu} dS_\mu + \int_{S_4} \sqrt{-g}\, T^{\mu\nu} dS_\mu$$

$S_1 : \tau = \tau_1 \qquad dS_\mu = -\frac{J}{u^2} u_\mu \, d\xi \, dw^1 dw^2$

$S_2 : \tau = \tau_2 \qquad dS_\mu = \frac{J}{u^2} u_\mu \, d\xi \, dw^1 dw^2$

$S_3: \xi = \xi_1 \qquad dS_\mu = J \frac{v_\mu}{u^2} \, d\tau \, d\omega^1 d\omega^2$

$S_4: \xi = \xi_2 \qquad dS_\mu = \frac{J \, v_\mu}{u^2} \, d\tau \, d\omega^1 d\omega^2$

$\Rightarrow$

$$\iint_\Sigma \partial_\mu (T^{\mu\nu} \sqrt{-g}) \, d\Sigma = \int_{\xi_1}^{\xi_2} u^\nu \Big|_{\tau=\tau_1} d\xi - \int_{\xi_1}^{\xi_2} u^\nu \Big|_{\tau=\tau_2} d\xi$$

$$- \int_{\tau_1}^{\tau_2} v^\nu \Big|_{\xi=\xi_1} d\tau + \int_{\tau_1}^{\tau_2} v^\nu \Big|_{\xi=\xi_2} d\tau$$

$$= \int_{\xi_1}^{\xi_2}\int_{\tau_1}^{\tau_2} \left(\frac{\partial}{\partial \tau} u^\nu - \frac{\partial}{\partial \xi} v^\nu \right) d\tau \, d\xi$$

$$= \iint \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} \right) z^\nu \, d\tau \, d\xi$$

$$\iint_\Sigma \Gamma^\nu_{\alpha\beta} T^{\mu\nu} \sqrt{-g} \, d\Sigma = \iint_\Sigma \Gamma^\nu_{\alpha\beta} \cdot \int H^{\alpha\beta} \delta^4(x-z) \, d\tau \, d\xi \, d\Sigma$$

$$= \iint H^{\alpha\beta} \Gamma^\nu_{\alpha\beta} \, d\tau \, d\xi$$

$$\Rightarrow \left(\frac{\partial^2}{\partial \tau^2} - \frac{\partial^2}{\partial \xi^2} \right) z^\mu + \Gamma^\mu_{\alpha\beta} H^{\alpha\beta} = 0$$

"string's equation of motion"

Field of Massless Closed Strings

Electromagnetic field of a point charge

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

where

$$A_\mu(x) = e\int_{s_i}^{s_f} ds\, \dot{z}'^{\mu}(s)\, \delta[(x - z'(s))^2]_{ret.}$$

is the Liénard-Wiechert potential of the point charge, $\dot{z}'^{\mu} = \frac{dz'^{\mu}}{ds}$

Similarly defining Kalb-Ramond potential

$$\phi_{\mu\nu}(x) = g\int_{\tau_i}^{\tau_f}\int_0^{\ell} \sigma_{\mu\nu}(\tau,\xi)\, \delta[(x - z(\xi,\tau))^2]_{ret}$$

with $\partial_\mu \phi^{\mu\nu} = 0$. The field of this potential is

$$F_{\mu\nu\alpha} = \partial_\mu \phi_{\nu\rho} + \partial_\nu \phi_{\rho\mu} + \partial_\rho \phi_{\mu\nu}$$

Invariant under the gauge transformation

$$\phi_{\mu\nu} \to \phi_{\mu\nu} + \partial_\mu \Lambda_\nu - \partial_\nu \Lambda_\mu .$$

The field equations

$$\partial_\mu F^{\mu\nu\alpha} = 0 \qquad \text{or} \quad \Box \phi_{\mu\nu} = 0$$

(with proper gauge choice).

Energy momentum tensor

$$T_{\mu\nu} = F_{\mu\alpha\beta} F_\nu{}^{\alpha\beta} - \frac{1}{6} F^2 g_{\mu\nu} .$$

with $F^2 = F^{\alpha\beta\gamma} F_{\alpha\beta\gamma}$

Define now

$$F_{\mu\nu\alpha} = \epsilon_{\mu\nu\alpha\rho} F^\rho$$

$$\Rightarrow \quad F^2 = -6\, F_\mu F^\mu$$

$$F^{\mu\alpha\beta} F_{\nu\alpha\beta} = -2 (F^\alpha F_\alpha) \delta^\mu{}_\nu + 2 F^\mu F_\nu$$

$\Rightarrow$

$$T_{\mu\nu} = 2 F_\mu F_\nu - 2 F^\alpha F_\alpha g_{\mu\nu} + F^\alpha F_\alpha g_{\mu\nu}$$

$$= 2 F_\mu F_\nu - (F^\alpha F_\alpha) g_{\mu\nu}$$

F^μ is either null or space-like.

1) $T_{\mu\nu}$ looks like the energy-momentum tensor of perfect fluid distribution

2) closed string fluids in "stars" or in "cosmology"?

3) can contribute to the acceleration of the universe (negative pressure $-F^\alpha F_\alpha$)