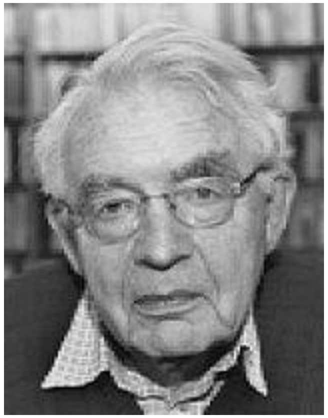
The Casimir Effect in Krein Space
(two parallel plates & a cylinder)

Outline

- A brief review of Casimir effect
- Krein space
- Casimir effect in Krein Space Quantization
- Some examples (*two parallel plates and a cylinder*)



H. Casimir (1909-2000)

Casimir effect

An attractive force between two *uncharged*, parallel conducting plates was predicted by Casimir in 1948,

this effect quickly received increasing attention

In 1996, this tiny force was experimentally measured,

S. K. Lamoreaux [Phys. Rev. Lett. **78**, 5 (1997)],

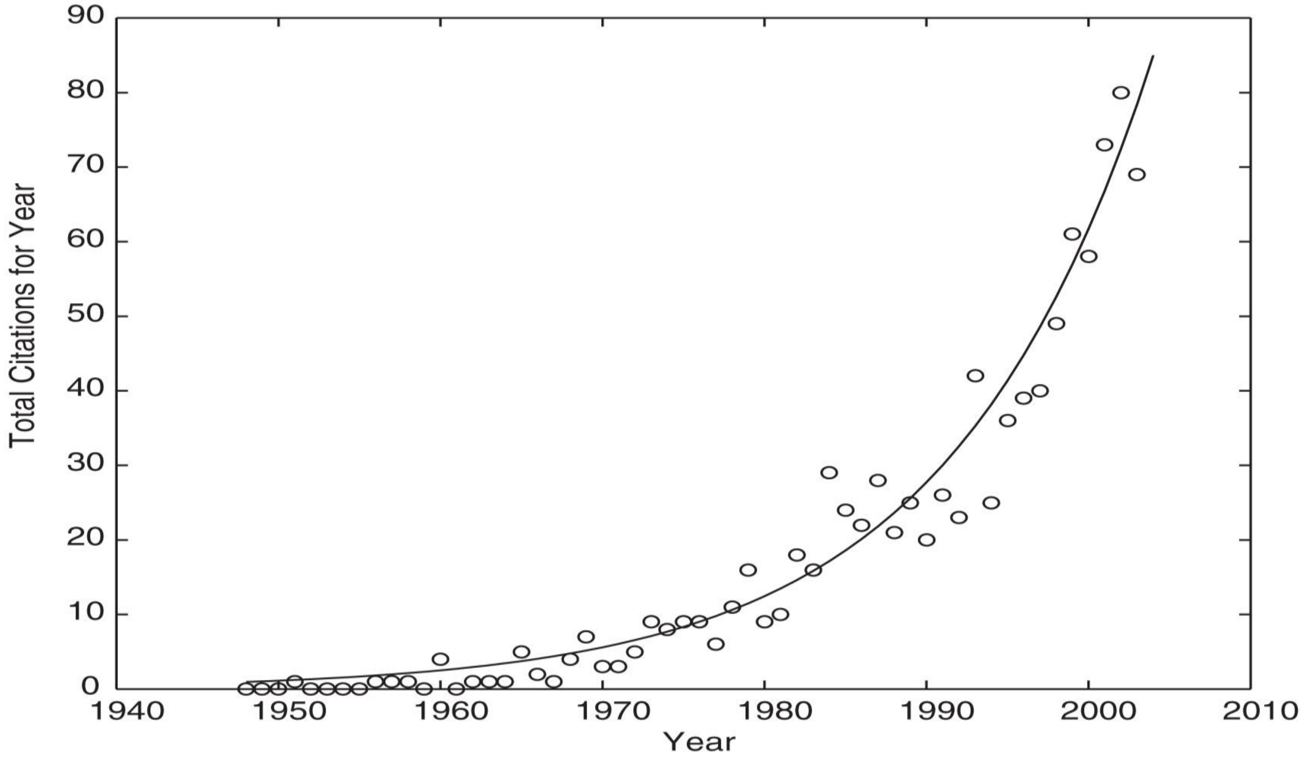


Figure 1. Number of citations per year of Casimir's 1948 paper. The time constant of exponential increase is about 12 years.

[Graph from D. Budker's powerpoint presentation]

- This is a pure quantum effect,

it can be regarded as the physical manifestation of the zero-point energy,

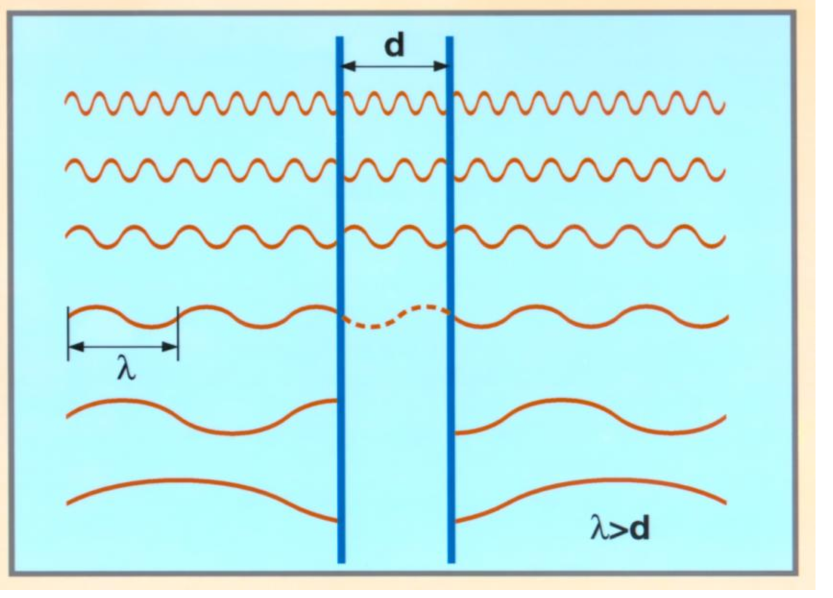
and:

its quantum nature can be understood by the zero-point fluctuation of the quantum fields

- In a simple way it is…

only those virtual photons
whose wavelengths satisfy
the following condition

$$d = n\frac{\lambda}{2}, \quad \text{n is integer,}$$



It is a boundary condition…

there exists an infinite number of virtual photons inside and outside, So, there are two infinities…

these cause a quantum pressure!!

in the ideal situation, at zero temperature for instance, there are no real photons …

So, it is only the vacuum, i.e., the ground state of QED which causes the plates to attract each other

M. Bordag, et al, Advances in the Casimir Effect, Oxford Pub. 2009

- One can find a lot of interesting papers about the Casimir effect in the literatures

Krein space

- In fact:

  Krein space is defined as a direct sum of an Hilbert space and an atni-Hilbert space (negative inner product space)

$$\mathcal{K} = \mathcal{H}_{+} \oplus \mathcal{H}_{-}$$

- Note that this space (K) is equipped with an indefinite inner product which means that some (`unphysical’) states have a negative norm.
- these ‘unphysical’ states do not propagate in the physical world;
- the direct consequence is that:
- the physical boundary conditions have no effect on these sates,

- and the mean values of observables are computed on physical states and no negative energy appears b.pdf

T. Garidi, et.al, J.Phys. A38 (2005) 245

Krein space vs. Hilbert space: The advantages

- This method has already been successfully applied to covariant quantization of massless minimally coupled scalar field in de Sitter space,
- it means:
- the covariant quantization should be carried out in Krein Space,

J-P Gazeau, et.al, Class. Quant. Grav. 17 (2000) 1415

- it was shown that within this method of quantization, the one-loop effective action for QED remains automatically finite

moreover, the vacuum energy of the free field vanishes without any need of reordering the terms $< 0|T^{00}|0 >= 0$

- M.V. Takook, Phys. Lett. B 704, 326 (2011)

M.V. Takook Int.J.Mod.Phys. E11 (2002) 509-518,

- Ibid Int.J.Mod.Phys. E14 (2005) 219-224 .

Examples:
The Scalar field with B.C.

Klein-Gordon equation:

$$(\Box - m^2)\phi = 0,$$

two sets of solutions:

$$u_P(\vec{k}, t, \vec{x}) = \frac{e^{i\vec{k}\cdot\vec{x} - iwt}}{\sqrt{(2\pi)2w}}, \text{ and } u_N(\vec{k}, t, \vec{x}) = \frac{e^{-i\vec{k}\cdot\vec{x} + iwt}}{\sqrt{(2\pi)2w}},$$

- According to the following inner product:

$$\langle \phi_1, \phi_2 \rangle = i \int_{t=const} dx(\phi_1^* \partial_t \phi_2 - \phi_2 \partial_t \phi_1^*),$$

- We have:

$$\langle u_P(\vec{k}, \vec{x}, t), u_P(\vec{k}', \vec{x}, t) \rangle = \delta(\vec{k} - \vec{k}'),$$
$$\langle u_N(\vec{k}, \vec{x}, t), u_N(\vec{k}', \vec{x}, t) \rangle = -\delta(\vec{k} - \vec{k}'),$$
$$\langle u_P(\vec{k}, \vec{x}, t), u_N(\vec{k}', \vec{x}, t) \rangle = 0.$$

These modes will be used in the definition of the quantum fields in Krein space

Hilbert space calculations, 1+1 dim.s:

- for two parallel plates, by imposing the boundary condition namely

$$u_p(k,0,t) = u_p(k,a,t) = 0,$$

- we obtain:

$$u_p(k_N,x,t) = (\frac{1}{a\omega_N})^{1/2} e^{-i\omega_N t} sin k_N x,$$

- where

$$\omega_N = (m^2 + k_N^2)^{1/2},\ k_N = \frac{N\pi}{a},\ N = 1,2,3,\ldots$$

- Total vacuum energy between the plates becomes:

$$E_0(a) = \int_0^a \langle 0|T_{00}|0\rangle dx = \frac{1}{2}\sum_{N=1}^{\infty} \omega_N.$$

- In QFT we are familiar with such kind of infinities. After regularization and renormalizion, the finite value is obtained.

- as

$$E_0^{Ren} = -\frac{\pi}{24a},$$

- This is the Casimir energy
- accordingly the Casimir force reads

$$F(a) = -\frac{\partial}{\partial a} E_0^{Ren} = -\frac{\pi}{24a^2},$$

Field operators in Krein Space

- In Krein space the field operator is defined by:

$$\phi(t,x) = \phi_P(t,x) + \phi_N(t,x)$$

- where:

$$\phi_P(t,\vec{x}) = \int d\vec{k}[a(\vec{k})u_P(\vec{k},t,\vec{x}) + a^\dagger(\vec{k})u_P^*(\vec{k},t,\vec{x})],$$
$$\phi_N(t,\vec{x}) = \int d\vec{k}[b(\vec{k})u_N(\vec{k},t,\vec{x}) + b^\dagger(\vec{k})u_N^*(\vec{k},t,\vec{x})].$$

- And after imposing the B.D. the quantum field reads as

$$\phi(t,x) = \sum_N \Big(a_N u_p(k_N,x,t) + a_N^\dagger u_p^*(k_N,x,t) \Big)$$
$$+ \int dk \Big(b(k) u_n(k,t,x) + b^\dagger(k) u_n^*(k,t,x) \Big).$$

- (note that the boundary conditions are imposed on the positive norm states)

- The direct result is c.pdf

$$E_0^{Kre}(a) = \int_0^a \langle\Omega|T_{00}^{Kre}|\Omega\rangle dx = -\frac{\pi}{24a},$$

- And

$$F(a) = -\frac{\partial(E_0^{Kre}(a))}{\partial a} = -\frac{\pi}{24}\frac{1}{a^2}$$

M.V. Takook et. al , Phys. Lett. B 6, 40 (2006) 48

Casimir effect in a cylinder

- Hilbert space Cal.s
- For a 1+1 dimensional cylinder where the spatial line element points x and x+L are identified, and L is the periodicity length. Imposing the periodic boundary condition:

$$u_k(t,x) = u_k(t,x+nL)$$

- one obtains:

$$u_k = \frac{1}{\sqrt{2L\omega}} e^{i(kx-\omega t)},$$

- where

$$k = \frac{2\pi n}{L}, \;\; n = 0, \pm 1, \pm 2, \cdots$$

- Within the usual calculation we have

$$\langle 0_L | T_{00} | 0_L \rangle = \frac{1}{2L} \sum_{n=-\infty}^{\infty} |k| = \frac{2\pi}{L^2} \sum_{n=0}^{\infty} n,$$

- which clearly results in an infinite vacuum energy, so we need to renormalize
- …

$$< 0|T^{00}|0 > := -\frac{\pi}{6L^2}$$

- N.D. Birrell and P.C.W. Davies, Cambridge University Press (1982) Quantum Fields in Curved Space, page 93

- Let us see what happens if we use Krein space instead of Hillbert space:

$$\langle\Omega|T_{00}^{Krein}|\Omega\rangle = \Big(\frac{1}{L}\sum_{n=0}^{\infty}\frac{2\pi n}{L} - \frac{1}{2\pi}\int_0^{\infty} k\,dk\Big),$$

- After using the Abel-Plana d.pdf formula, and doing some calculations, one ontains:

$$\langle\Omega|T_{00}^{Krein}|\Omega\rangle = \frac{2\pi}{L^2}\Big(-\frac{1}{12}\Big) = -\frac{\pi}{6L^2},$$

This is exactly the same as other related works

M. Reza Tanhayi, et.al, "The casimir effect in Krein Space for cylinder, Torus and Sphere" in preparation

Conclusion:

- In our recent work, we used Krein space to calculate the Casimir effect of a sphere,
- we obtained the exact result.
- It seems that if one uses Krein space, some infinities will be disappeared, so…
- quantization procedure can be carried out in this way somehow easier

- *Thank you*