Higgs Mechanism for New Massive Gravity and Weyl-Invariant Extensions of Higher-Derivative Theories

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Problem of the Mass of Gravitons

- Giving mass to graviton is a subtle problem.
- Providing a Fierz-Pauli type hard mass, $m^2(h_{\mu\nu}^2 - h^2)$, to graviton has problems:
  - Gauge invariance is lost.
  - Boulware-Deser ghost appears at the nonlinear level.
  - van Dam-Veltman-Zakharov discontinuity: tree-level scattering potentials for $m^2 = 0$ and $m^2 \to 0$ are different.

Main Question

- **Is there any Higgs-type mechanism for the spin-2 graviton?**

What is this talk about?

- In this talk, such a Higgs-type mechanism is presented for massive spin-2 graviton by:
  - First, constructing the Weyl-invariant new massive gravity (NMG) and then showing that the vacua of the theory breaks Weyl symmetry,
  - Symmetry breaking depends on the background (assuming one does not add any explicit symmetry breaking term by hand).
  - Spectrum and stability of the theory is discussed.
- Besides, Weyl-invariant form of other Higher-derivative theories are also presented.

Quadratic Curvature Gravity 4D

- Einstein's gravity $\frac{1}{\kappa}R$ is not renormalizable. ('t Hooft and Veltman, 1974; Deser and van Nieuwenhuizen, 1974)
- Quadratic curvature theory $\frac{1}{\kappa}R + \alpha R^2 + \beta R_{\mu\nu}^2$ is renormalizable, (Stelle, 1977)
- BUT not unitary: unitarity of massless spin-2 and massive spin-2 modes are in conflict.

Quadratic Curvature Gravity 3D

- Action of the generic quadratic curvature theory in $(1+2)$-dimensional spacetime with arbitrary parameters κ,α, and β:

$$I = \int d^3x \sqrt{-g} \left(\frac{1}{\kappa} R + \alpha R^2 + \beta R_{\mu\nu}^2 \right).$$

- Expectation: this theory is superrenormalizable, since its $4D$ cousin is renormalizable.

- Einstein-Hilbert action does not propagate massless spin-2 mode in 3D. There may be a cure for unitarity.

New Massive Gravity - I

- Yes, there is a cure. $8\alpha + 3\beta = 0$ combination yields a unitary theory with the action

$$I_{NMG} = \frac{1}{k^2} \int d^3x \sqrt{-g} \left[-R + \frac{1}{m_g^2} \left(R_{\mu\nu}^2 - \frac{3}{8} R^2 \right) \right],$$

where $\frac{1}{\kappa} = -\frac{1}{k^2}$, "wrong" sign Einstein-Hilbert term as in topologically massive gravity (Bergshoeff, Hohm and Townsend, PRL, 2009).

New Massive Gravity - II

- NMG describes nonlinear, generally covariant extension of the Fierz-Pauli massive gravity.
- At linearized level, theory describes a massive graviton with 2 degrees of freedom both around its flat and (anti)-de Sitter vacua.
- The theory is unitary at tree-level for certain choices and ranges of parameters in flat and (A)dS backgrounds [Bergshoeff *et al* (2009), Tekin *et al* (2009), Deser (2009), Nakasone *et al* (2009)].
- In AdS, it is unitary either in the bulk or on the boundary (Bergshoeff *et al* PRD, 2010).
- Born-Infeld extension of NMG (BINMG) was proposed by Gullu, Sisman and Tekin (CQG, 2010).

Weyl-Invariance of Scalar Field

- Simplest example is the kinetic part of the scalar field action

$$S_\Phi = -\frac{1}{2}\int d^n x \sqrt{-g}\,\partial_\mu \Phi \partial_\nu \Phi g^{\mu\nu}.$$

- To make the action Weyl-invariant, one should have invariance under

$$g_{\mu\nu} \to g'_{\mu\nu} = e^{2\lambda(x)} g_{\mu\nu}, \qquad \Phi \to \Phi' = e^{-\frac{(n-2)}{2}\lambda(x)}\Phi,$$

- Derivatives should be replaced with (real) gauge covariant derivative

$$\mathscr{D}_\mu \Phi = \partial_\mu \Phi - \frac{n-2}{2} A_\mu \Phi,$$

where A_μ is Weyl's gauge field

$$A_\mu \to A'_\mu = A_\mu - \partial_\mu \lambda(x).$$

- Then, conformal transformation of gauge covariant derivative becomes

$$(\mathscr{D}_\mu \Phi)' = e^{-\frac{(n-2)}{2}\lambda(x)} \mathscr{D}_\mu \Phi.$$

Weyl-Invariance of Vector Field

- Field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is gauge invariant.
- But, Maxwell-type action needs a compensating Weyl scalar

$$S_{A^\mu} = -\frac{1}{2}\int d^n x \sqrt{-g}\, \Phi^{\frac{2(n-4)}{n-2}} F_{\mu\nu}F^{\mu\nu}.$$

Weyl-Invariance of Tensor Field - I

- In order to implement the Weyl invariance into gravity, one should define a Weyl-invariant connection with the help of regular Christoffel connection and gauge field:

$$\tilde{\Gamma}^{\lambda}_{\mu\nu} = \frac{1}{2} g^{\lambda\sigma}\Big(\mathscr{D}_\mu g_{\sigma\nu} + \mathscr{D}_\nu g_{\mu\sigma} - \mathscr{D}_\sigma g_{\mu\nu}\Big),$$

  where

$$\mathscr{D}_\mu g_{\alpha\beta} = \partial_\mu g_{\alpha\beta} + 2A_\mu g_{\alpha\beta}, \qquad \left(\mathscr{D}_\mu g_{\alpha\beta}\right)' = e^{2\lambda(x)}\mathscr{D}_\mu g_{\alpha\beta}.$$

- Weyl-invariant "Riemann tensor" becomes

$$\begin{aligned}\tilde{R}^{\mu}{}_{\nu\rho\sigma}[g,A] &= \partial_\rho \tilde{\Gamma}^{\mu}_{\nu\sigma} - \partial_\sigma \tilde{\Gamma}^{\mu}_{\nu\rho} + \tilde{\Gamma}^{\mu}_{\lambda\rho}\tilde{\Gamma}^{\lambda}_{\nu\sigma} - \tilde{\Gamma}^{\mu}_{\lambda\sigma}\tilde{\Gamma}^{\lambda}_{\nu\rho} \\ &= R^{\mu}{}_{\nu\rho\sigma} + \delta^{\mu}{}_{\nu}F_{\rho\sigma} + 2\delta^{\mu}{}_{[\sigma}\nabla_{\rho]}A_\nu + 2g_{\nu[\rho}\nabla_{\sigma]}A^\mu \\ &\quad + 2A_{[\sigma}\delta_{\rho]}{}^{\mu}A_\nu + 2g_{\nu[\sigma}A_{\rho]}A^\mu + 2g_{\nu[\rho}\delta_{\sigma]}{}^{\mu}A^2.\end{aligned}$$

Weyl-Invariance of Tensor Field - II

- Weyl-invariant Ricci tensor becomes

$$\begin{aligned}\tilde{R}_{\nu\sigma}[g,A] &= \tilde{R}^{\mu}{}_{\nu\mu\sigma}[g,A] \\ &= R_{\nu\sigma} + F_{\nu\sigma} - (n-2)\Big[\nabla_\sigma A_\nu - A_\nu A_\sigma + A^2 g_{\nu\sigma}\Big] \\ &\quad - g_{\nu\sigma}\nabla\cdot A.\end{aligned}$$

- Finally, Ricci scalar can be computed as

$$\tilde{R}[g,A] = R - 2(n-1)\nabla\cdot A - (n-1)(n-2)A^2,$$

which is *not* Weyl invariant, but transforms as

$$(\tilde{R}[g,A])' = e^{-2\lambda}\tilde{R}[g,A].$$

Weyl-Invariance of Tensor Field - III

- Since the curvature tensor is not Weyl-invariant, the Weyl-invariant Einstein-Hilbert action requires a compensating Weyl scalar:

$$S=\int d^nx\sqrt{-g}\Phi^2\tilde{R}[g,A]=\int d^nx\sqrt{-g}\Phi^2\left(R-2(n-1)\nabla\cdot A-(n-1)(n-2)A^2\right).$$

- One can eliminate the gauge field by using its field equation $A_\mu=\frac{2}{n-2}\partial_\mu\ln\Phi$, to get "the conformally coupled scalar-tensor" theory

$$S=\int d^nx\sqrt{-g}\left(\Phi^2R+4\frac{(n-1)}{n-2}\partial_\mu\Phi\partial^\mu\Phi\right).$$

- For particular constant value of the scalar field, this action reduces to general relativity without cosmological constant.

- To introduce cosmological constant, a Weyl-invariant potential should be added to the scalar field action:

$$S_\Phi=-\frac{1}{2}\int d^nx\sqrt{-g}\left(\mathscr{D}_\mu\Phi\mathscr{D}^\mu\Phi+\nu\Phi^{\frac{2n}{n-2}}\right),$$

where $\nu\geq0$ is a dimensionless coupling constant.

Weyl-Invariant extension of NMG - I

- Using the tools developed, the Weyl-invariant extension of NMG can be written as

$$\tilde{S}_{NMG} = \int d^3x\sqrt{-g}\left[\sigma\Phi^2\tilde{R} + \Phi^{-2}\left(\tilde{R}^2_{\mu\nu} - \frac{3}{8}\tilde{R}^2\right)\right] + S_\Phi + S_{A_\mu},$$

where $\sigma = \pm 1$ (Dengiz, Tekin, PRD 2011).

- Here, S_Φ and S_{A_μ} are the three dimensional forms of the Weyl-invariant scalar and gauge field actions described before:

$$S_\Phi = -\frac{1}{2}\int d^3x\sqrt{-g}\left\{\left(\partial_\mu\Phi - \frac{1}{2}A_\mu\Phi\right)^2 + \nu\Phi^6\right\},$$

$$S_{A_\mu} = \beta\int d^3x\sqrt{-g}\,\Phi^{-2}F^2_{\mu\nu},$$

where β parameter is introduced and will be relevant in unitarity discussion.

Weyl-Invariant extension of NMG - II

- The explicit form of Weyl-invariant NMG

$$\tilde{S}_{NMG} = \int d^3x\sqrt{-g}\Big\{\sigma\Phi^2\Big(R - 4\nabla\cdot A - 2A^2\Big) + \Phi^{-2}\Big[R_{\mu\nu}^2 - \frac{3}{8}R^2 - 2R^{\mu\nu}\nabla_\mu A_\nu + 2R^{\mu\nu}A_\mu A_\nu + R\nabla\cdot A - \frac{1}{2}RA^2 + 2F_{\mu\nu}^2 + (\nabla_\mu A_\nu)^2 - 2A_\mu A_\nu \nabla^\mu A^\nu - (\nabla\cdot A)^2 + \frac{1}{2}A^4\Big]\Big\} + S_\Phi + S_{A_\mu}.$$

- Formally, for $A_\mu = 0$, $\Phi = \sqrt{m}$ and $\nu = 2\lambda \Rightarrow$ NMG with a fixed coupling $\kappa = m^{-1/2}$.

One Scale in the Theory

- If the scalar field freezes, the Newton's constant is related to the mass of graviton.
- Question: Does vacuum break Weyl symmetry?

Field Equations - I

- Variation with respect to $g^{\mu\nu}$:

$$
\begin{aligned}
&\sigma\Phi^2 G_{\mu\nu} + \sigma g_{\mu\nu}\Box\Phi^2 - \sigma\nabla_\mu\nabla_\nu\Phi^2 - 4\sigma\Phi^2\nabla_\mu A_\nu + 2\sigma g_{\mu\nu}\Phi^2\nabla\cdot A - 2\sigma\Phi^2 A_\mu A_\nu \\
&+ \sigma g_{\mu\nu}\Phi^2 A^2 + 2\Phi^{-2}[R_{\mu\sigma\nu\alpha} - \frac{1}{4}g_{\mu\nu}R_{\sigma\alpha}]R^{\sigma\alpha} + \Box(\Phi^{-2}G_{\mu\nu}) + \frac{1}{4}[g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu]\Phi^{-2}R \\
&+ g_{\mu\nu}G^{\sigma\alpha}\nabla_\sigma\nabla_\alpha\Phi^{-2} - 2G^\sigma{}_\nu\nabla_\sigma\nabla_\mu\Phi^{-2} - 2(\nabla_\mu G^\sigma{}_\nu)(\nabla_\sigma\Phi^{-2}) + \frac{3}{16}g_{\mu\nu}\Phi^{-2}R^2 \\
&- \frac{3}{4}\Phi^{-2}RR_{\mu\nu} + g_{\mu\nu}\Phi^{-2}R_{\alpha\beta}\nabla^\alpha A^\beta - 2\Phi^{-2}R_{\alpha\nu}\nabla_\mu A^\alpha - 2\Phi^{-2}R_{\beta\mu}\nabla^\beta A_\nu - \Box(\Phi^{-2}\nabla_\mu A_\nu) \\
&- g_{\mu\nu}\nabla_\beta\nabla_\alpha(\Phi^{-2}\nabla^\alpha A^\beta) + \nabla_\alpha\nabla_\nu(\Phi^{-2}\nabla^\alpha A_\mu) + \nabla_\beta\nabla_\nu(\Phi^{-2}\nabla_\mu A^\beta) - g_{\mu\nu}\Phi^{-2}R^{\alpha\beta}A_\alpha A_\beta \\
&+ 4\Phi^{-2}R_{\alpha\nu}A_\mu A^\alpha + \Box(\Phi^{-2}A_\mu A_\nu) - 2\nabla^\alpha\nabla_\nu(\Phi^{-2}A_\alpha A_\mu) + g_{\mu\nu}\nabla^\alpha\nabla^\beta(\Phi^{-2}A_\alpha A_\beta) \\
&+ \Phi^{-2}G_{\mu\nu}\nabla\cdot A + g_{\mu\nu}\Box(\Phi^{-2}\nabla\cdot A) - \nabla_\mu\nabla_\nu(\Phi^{-2}\nabla\cdot A) + \Phi^{-2}R\nabla_\mu A_\nu - \frac{1}{2}\Phi^{-2}G_{\mu\nu}A^2 \\
&- \frac{1}{2}g_{\mu\nu}\Box(\Phi^{-2}A^2) + \frac{1}{2}\nabla_\mu\nabla_\nu(\Phi^{-2}A^2) - \frac{1}{2}\Phi^{-2}RA_\mu A_\nu - \Phi^{-2}[g_{\mu\nu}F^2_{\alpha\beta} + 4F_\nu{}^\alpha F_{\alpha\mu}] \\
&- \frac{1}{2}g_{\mu\nu}\Phi^{-2}(\nabla_\alpha A_\beta)^2 + \Phi^{-2}\nabla_\mu A_\alpha\nabla_\nu A^\alpha + \Phi^{-2}\nabla_\beta A_\nu\nabla^\beta A_\mu + g_{\mu\nu}\Phi^{-2}A^\alpha A^\beta\nabla_\alpha A_\beta \\
&- 2\Phi^{-2}A_\nu A^\alpha\nabla_\mu A_\alpha - 2\Phi^{-2}A_\mu A^\beta\nabla_\beta A_\nu + \frac{1}{2}g_{\mu\nu}(\nabla\cdot A)^2 - 2(\nabla\cdot A)\nabla_\mu A_\nu \\
&- \frac{1}{4}g_{\mu\nu}\Phi^{-2}A^4 + \Phi^{-2}A_\mu A_\nu A^2 = -\frac{1}{\sqrt{-g}}\frac{\delta S_\Phi}{\delta g^{\mu\nu}}.
\end{aligned}
$$

Field Equations - II

- Variation with respect to Φ:

$$2\sigma\Phi\Big(R-4\nabla\cdot A-2A^2\Big)-2\Phi^{-3}\Big[R_{\mu\nu}^2-\frac{3}{8}R^2$$
$$-2R^{\mu\nu}\nabla_\mu A_\nu+2R^{\mu\nu}A_\mu A_\nu+R\nabla\cdot A-\frac{1}{2}RA^2$$
$$+2F_{\mu\nu}^2+(\nabla_\mu A_\nu)^2-2A_\mu A_\nu\nabla^\mu A^\nu-(\nabla\cdot A)^2$$
$$+\frac{1}{2}A^4\Big]=-\frac{1}{\sqrt{-g}}\frac{\delta S_\Phi}{\delta\Phi}.$$

- Variation with respect to the Weyl gauge-field:

$$-4\nabla_\mu\Phi^2+4\Phi^2A_\mu+2R^\nu{}_\mu\nabla_\nu\Phi^{-2}+4R_{\mu\nu}A^\nu$$
$$-R\nabla_\mu\Phi^{-2}-\Phi^{-2}RA_\mu+8\nabla^\nu(\Phi^{-2}\nabla_\mu A_\nu)$$
$$-10\nabla^\nu(\Phi^{-2}\nabla_\nu A_\mu)+2\nabla_\alpha(\Phi^{-2}A^\alpha A_\mu)$$
$$-2\Phi^{-2}(\nabla_\mu A_\nu)A^\nu-2\Phi^{-2}(\nabla_\nu A_\mu)A^\nu$$
$$+2\nabla_\mu(\Phi^{-2}\nabla\cdot A)+2A_\mu A^2=-\frac{1}{\sqrt{-g}}\frac{\delta S_\Phi}{\delta A^\mu}.$$

Ansatz and Vacuum Field Equation

- In order to avoid breaking local Lorentz invariance of the vacuum, let us set $F_{\mu\nu} = 0$ and choose the gauge for which $A_\mu = 0$. Let,

$$\Phi \equiv \sqrt{m}, \qquad R_{\mu\nu} = 2\Lambda g_{\mu\nu}.$$

- The gauge field equation is automatically satisfied and the other two field equations give the same equation

$$\nu\, m^4 - 4\sigma m^2 \Lambda - \Lambda^2 = 0.$$

(A)dS Background - I

- 1^{st} **case: Suppose that Λ is known.** Then, one gets

$$m_{\pm}^2 = \frac{2\sigma\Lambda}{\nu} \pm \frac{|\Lambda|}{\nu}\sqrt{4\sigma^2 + \nu}.$$

  Since, $\kappa = m^{-1/2} > 0$, the negative root is *not* allowed for both sign of Λ.

- Mass of the graviton is

$$M_g^2 = -\sigma m_+^2 + \frac{\Lambda}{2},$$

  which need to satisfy the Higuchi bound $M_g^2 \geq \Lambda > 0$ in dS and the Breitenlohner-Freedman bound $M_g^2 \geq \Lambda$ in AdS. Thus, one has

$$-4\text{sign}(\Lambda) - 2\sigma\sqrt{4+\nu} \geq \text{sign}(\Lambda)\nu.$$

- For $\Lambda > 0$, the theory is not unitary. . For $\Lambda < 0$, both signs of σ are allowed.

(A)dS Background - II

- 2^{nd} **Case: Suppose that the vacuum expectation value of the scalar field is known.** Then, the cosmological constant can be evaluated as

$$\Lambda_{\pm} = m^2 \Big[-2\sigma \pm \sqrt{4+\nu} \Big].$$

For $\nu > 0$, same analysis in the 1^{st}case holds.

- For the case $\nu = 0$: $\Lambda = -4\sigma m^2$
  - in de-Sitter, $\sigma = -1$ is allowed; M_g^2 becomes $-3\sigma m^2$ but *Higuchi bound* is not satisfied $\Rightarrow$ the theory is *not unitary*.
  - in Anti-de Sitter, $\sigma = +1$ is allowed and Breitenlohner-Freedman bound is satisfied $\Rightarrow$ the theory is *unitary*.

Flat Background - Back to Tan, Tekin, Hosotani (1996, 97)

- **For the case of flat background**:
  - Around the flat background, $\Lambda = 0 \Rightarrow m = 0$; the Weyl symmetry of the Lagrangian is not broken by the vacuum solution. One way to break symmetry is to add an explicit mass term to the Lagrangian.
  - Alternatively, one can check whether the radiative corrections do break the symmetry as was shown by Coleman and Weinberg in the massless Φ^4 theory in *four dimensions*.
  - In *three dimensions*, the computation for $\nu\Phi^6$ theory was carried out in *P.N. Tan, B. Tekin.* and *Y. Hosotani* (1996 and 1997). At two-loop level, the effective scalar potential was found as

$$V_{eff} = \nu(\mu)\Phi^6 + \frac{7\hbar^2}{120\pi^2}\nu(\mu)^2\Phi^6\left(\ln\frac{\Phi^4}{\mu^2} - \frac{49}{5}\right).$$

  - Thus, it is obvious that the minimum of the potential is away from $\Phi = 0$ and the Weyl symmetry is broken as we desire.

Quadratic Action in Fluctuations - I

- In order to study spectrum and stability, one needs the quadratic action in fluctuations around the vacuum:

$$\Phi = \sqrt{m} + \tau \Phi_L, \qquad A_\mu = \tau A^L_\mu, \qquad g_{\mu\nu} = \bar{g}_{\mu\nu} + \tau h_{\mu\nu}.$$

- Quadratic action in fluctuations (Tanhayi, Dengiz, Tekin, 2011):

$$\begin{aligned} S^{(2)}_{WNMG} = \int d^3x \sqrt{-\bar{g}} \Big\{ & -\frac{1}{2}(\partial_\mu \Phi^L)^2 + \Big(6\sigma\Lambda - \frac{9\Lambda^2}{2m^2} - \frac{15\nu m^2}{2}\Big)\Phi^2_L \\ & + \frac{2\beta+5}{2m}(F^L_{\mu\nu})^2 - \Big(2\sigma m + \frac{\Lambda}{m} + \frac{m}{8}\Big)A^2_L - \frac{1}{m}(\bar{\nabla}\cdot A^L)^2 \\ & + \frac{1}{m}(\mathcal{G}^L_{\mu\nu})^2 - \Big(\frac{\sigma m}{2} - \frac{\Lambda}{4m}\Big) h^{\mu\nu}\mathcal{G}^L_{\mu\nu} - \frac{1}{8m}R^2_L \\ & + \Big(2\sigma\sqrt{m} + \frac{\Lambda}{m\sqrt{m}}\Big)\Phi^L R^L - \Big(8\sigma\sqrt{m} + \frac{4\Lambda}{m\sqrt{m}} + \frac{\sqrt{m}}{2}\Big)\Phi^L \bar{\nabla}\cdot A^L \Big\}. \end{aligned}$$

- Here, $\mathcal{G}^L_{\mu\nu}$ is the linearized Einstein tensor, R_L is the linearized curvature scalar:

$$\mathcal{G}^L_{\mu\nu} = R^L_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}R_L - 2\Lambda h_{\mu\nu}, \qquad R_L = \nabla_\alpha\nabla_\beta h^{\alpha\beta} - \Box h - 2\Lambda h,$$

$$R^L_{\mu\nu} = \frac{1}{2}\left(\nabla^\sigma\nabla_\mu h_{\nu\sigma} + \nabla^\sigma\nabla_\nu h_{\mu\sigma} - \Box h_{\mu\nu} - \nabla_\mu\nabla_\nu h\right).$$

Quadratic Action in Fluctuations - II

- To have a non-ghost and canonically normalized (that is $-\frac{1}{4}$) kinetic term for the Maxwell field, one should set $\beta = -\frac{11}{4}$.
- Fields are coupled as demanded by conformal invariance.

Decoupling

- One needs to redefine the tensor field as

$$h_{\mu\nu} \equiv \widetilde{h}_{\mu\nu} - \frac{4}{\sqrt{m}} \bar{g}_{\mu\nu} \Phi_L.$$

- We can choose a Lorenz-like condition

$$\mathscr{D}_\mu A^\mu = \nabla \cdot A + A^2 = 0,$$

as a Weyl-invariant gauge-fixing condition.

- At the linear level, this gauge choice reduces to the background covariant Lorenz condition: $\bar{\nabla} \cdot A_L = 0$.

Decoupled Quadratic Action in Fluctuations

- After implementing the tensor redefinition and the gauge condition, quadratic action in fluctuations decouples as

$$\widetilde{S}_{WNMG} = \int d^3x\sqrt{-\bar{g}}\Big\{ -\frac{1}{2}\Big(16\sigma + \frac{8\Lambda}{m^2} + 1\Big)(\partial_\mu \Phi^L)^2 - \frac{1}{4m}(F^L_{\mu\nu})^2 - \Big(2\sigma m + \frac{\Lambda}{m} + \frac{m}{8}\Big)(A^L_\mu)^2 - \Big(\frac{\sigma m}{2} - \frac{\Lambda}{4m}\Big)\widetilde{h}^{\mu\nu}\widetilde{\mathscr{G}}^L_{\mu\nu} + \frac{1}{m}(\widetilde{\mathscr{G}}^L_{\mu\nu})^2 - \frac{1}{8m}\widetilde{R}^2_L\Big\}.$$

- Generically, spectrum of the theory involves massless spin-0, massive spin-1, massive spin-2 fields. Masses of the fields:

$$M_A^2 = (4\sigma + \frac{1}{4})m^2 + 2\Lambda, \qquad M_g^2 = -\sigma m^2 + \frac{\Lambda}{2},$$

where cosmological constant is

$$\Lambda_\pm = m^2\Big[-2\sigma \pm \sqrt{4+\nu}\Big].$$

Unitarity Conditions

- To have a non-ghost kinetic term for the massless scalar field:

$$16\sigma + \frac{8\Lambda}{m^2} + 1 \geq 0.$$

- To have a non-tachyonic mass term for the spin-1 field:

$$M_A^2 \geq 0,$$

which is exactly equal to previous condition.

- Unitarity of the massive spin-2 theory depends whether one is dealing with an AdS ($\Lambda < 0$) or a dS ($\Lambda > 0$) background. For the AdS background, Breitenlohner-Freedman (BF) bound $M_g^2 \geq \Lambda$ must be satisfied, on the other hand for the dS background, Higuchi bound $M_g^2 \geq \Lambda > 0$ must be satisfied.

Parameter Regions of Unitarity - I

- **dS background (Λ_+):** Higuchi bound cannot be satisfied. Theory is not unitary in dS.

- **AdS background (Λ_-):** Theory is unitary in AdS with a massless spin-0, a massive spin-1 and a massive spin-2 field as long as

$$-\frac{1}{16} < \sigma \leq 0, \qquad 0 < \nu \leq \frac{1}{64}(1-256\sigma^2),$$
$$0 < \sigma < \frac{1}{16}, \qquad 0 \leq \nu \leq \frac{1}{64}(1-256\sigma^2).$$

For

$$\sigma = \frac{1}{16}, \qquad \nu = 0, \qquad \Lambda_- = -\frac{m^2}{4},$$

the theory has a massless spin-1 field, a massive spin-2 field (with $M_g^2 = -\frac{3m^2}{16}$) (no scalar field).

Parameter Regions of Unitarity - II

- **Flat background:**
  - The theory becomes unitary when

    $$-\frac{1}{16} \leq \sigma \leq 0, \qquad \nu = 0,$$

    for which generically the theory has a massless spin-0, massive spin-1 and massive spin-2 fields.
  - For $\sigma = -\frac{1}{16}$, there is no scalar field, there is a massless gauge field and a massive spin-2 field with mass $M_g = \frac{m}{4}$.
  - For $\sigma = 0$, there is a massless spin-0, a massless spin-2 and a massive spin-1 field with $M_A = \frac{m}{2}$.

Weyl-Invariant Quadratic Gravity - I

- The generic Weyl-invariant quadratic gravity action is defined by the action

$$\tilde{S}_{quadratic} = \int d^n x \sqrt{-g}\, \Phi^{\frac{2(n-4)}{n-2}} \left[\alpha \tilde{R}^2 + \beta \tilde{R}^2_{\mu\nu} + \gamma \tilde{R}^2_{\mu\nu\rho\sigma} \right],$$

where the explicit form of the curvature square terms read as

$$\begin{aligned} \tilde{R}^2 =& R^2 - 4(n-1)R(\nabla\cdot A) - 2(n-1)(n-2)RA^2 \\ &+ 4(n-1)^2(\nabla\cdot A)^2 + 4(n-1)^2(n-2)A^2(\nabla\cdot A) \\ &+ (n-1)^2(n-2)^2A^4, \end{aligned}$$

with the definitions $\nabla.A = \nabla_\mu A^\mu$, $A^2 = A_\mu A^\mu$ and $A^4 = A_\mu A^\mu A_\nu A^\nu$.

Weyl-Invariant Quadratic Gravity - II

- $$\begin{aligned}\tilde{R}^2_{\mu\nu} = R^2_{\mu\nu} &- 2(n-2)R^{\mu\nu}\nabla_\nu A_\mu - 2R(\nabla\cdot A) + 2(n-2)R^{\mu\nu}A_\mu A_\nu \\ &- 2(n-2)RA^2 + F^2_{\mu\nu} - 2(n-2)F^{\mu\nu}\nabla_\nu A_\mu \\ &+ (n-2)^2(\nabla_\nu A_\mu)^2 + (3n-4)(\nabla.A)^2 \\ &- 2(n-2)^2 A_\mu A_\nu \nabla^\mu A^\nu + (4n-6)(n-2)A^2(\nabla\cdot A) \\ &+ (n-2)^2(n-1)A^4.\end{aligned}$$

- $$\begin{aligned}\tilde{R}^2_{\mu\nu\rho\sigma} = R^2_{\mu\nu\rho\sigma} &- 8R^{\mu\nu}\nabla_\mu A_\nu + 8R^{\mu\nu}A_\mu A_\nu - 4RA^2 + nF^2_{\mu\nu} \\ &+ 4(n-2)(\nabla_\mu A_\nu)^2 + 4(\nabla\cdot A)^2 + 8(n-2)A^2(\nabla\cdot A) \\ &- 8(n-2)A_\mu A_\nu \nabla^\mu A^\nu + 2(n-1)(n-2)A^4.\end{aligned}$$

Gauss-Bonnet Combination

- By using these curvature square terms, one can study any Weyl-invariant quadratic theory. Particularly, n-dimensional Weyl-invariant Gauss-Bonnet combination can be easily written as

$$\begin{aligned}\tilde{R}^2_{\mu\nu\rho\sigma} - 4\tilde{R}^2_{\mu\nu} + \tilde{R}^2 = {} & R^2_{\mu\nu\rho\sigma} - 4R^2_{\mu\nu} + R^2 + 8(n-3)R^{\mu\nu}\nabla_\mu A_\nu - 8(n-3)R^{\mu\nu}A_\mu A_\nu \\ & - 2(n-3)(n-4)RA^2 - (3n-4)F^2_{\mu\nu} - 4(n-2)(n-3)(\nabla_\mu A_\nu)^2 \\ & + 4(n-2)(n-3)(\nabla\cdot A)^2 + 4(n-2)(n-3)^2A^2(\nabla\cdot A) \\ & + 8(n-2)(n-3)A_\mu A_\nu\nabla^\mu A^\nu - 4(n-3)R(\nabla\cdot A) \\ & + (n-1)(n-2)(n-3)(n-4)A^4,\end{aligned}$$

which for the specific case of $n=3$ reduces to Maxwell theory

$$\tilde{R}^2_{\mu\nu\rho\sigma} - 4\tilde{R}^2_{\mu\nu} + \tilde{R}^2 = -5F^2_{\mu\nu}.$$

Weyl-Invariant BINMG

- Weyl-invariant form of Born-Infeld extension of NMG is

$$S_{BINMG} = -4 \int d^3x \left[\sqrt{-\det\left(\Phi^4 g + \sigma \tilde{G}\right)} - (1 - \frac{\lambda}{2}) \sqrt{-\Phi^4 g} \right],$$

where $\tilde{G}_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R}$ is the Weyl-invariant Einstein tensor.

- Expansion of the determinant in terms of the traces yields

$$\sqrt{-\det\left(\Phi^4 g + \sigma \tilde{G}\right)} = \sqrt{-\det\left(\Phi^4 g\right)} \left(1 - \frac{1}{2} \Phi^{-4} \tilde{R}^{\mu\nu} \left[-g_{\mu\nu} + \Phi^{-4} \left(\tilde{R}_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \tilde{R} \right) \right. \right.$$
$$\left. \left. + \frac{2}{3} \Phi^{-8} \left(\tilde{R}_{\mu\rho} \tilde{R}^{\rho}{}_{\nu} - \frac{3}{4} \tilde{R} \tilde{R}_{\mu\nu} + \frac{1}{8} g_{\mu\nu} \tilde{R}^2 \right) \right] \right)^{1/2},$$

which is exact up to this point.

- From this expression, one can construct Weyl-invariant theories at any order in the curvature by doing a Taylor series expansion in the curvature.

Conclusions

- Weyl-invariant extension of NMG is a unitary theory generically describing a massive spin-2, a massive (or massless) spin-1 and a massless spin-0 fields around its AdS and flat vacua.
- The mere existence of an AdS background spontaneously breaks the conformal symmetry and provides mass to the spin-1 and spin-2 fields in analogy with the Higgs mechanism. Breaking of the conformal symmetry also fixes all the relevant couplings between the fields.
- In flat space, dimensionful parameter (that is the expectation value of the scalar field) comes from dimensional transmutation in the quantum theory and the conformal symmetry is broken at the two loop level via the Coleman-Weinberg mechanism.
- Weyl-invariant version of NMG seems to be the only known toy model where a graviton mass is generated by the breaking of a symmetry in such a way that the resultant mass has a non-linear, fully covariant, local extension in terms of quadratic curvature terms.